* Typos in the last lecture: $|0\ldots0\rangle = \sin\theta|\psi\rangle + \cos\theta|\psi_\perp\rangle$

A.A.: $R_0 R_\psi|0\ldots0\rangle = \sin(\theta(1+2))|\psi\rangle + \cos(\theta(1+2))|\psi_\perp\rangle$

$(R_0 R_\psi)^n|0\ldots0\rangle = \sin\theta(\theta(1+2n))|\psi\rangle + \cos(\theta(1+2n))|\psi_\perp\rangle$

7. Qubitization: Quantum Signal Proessing

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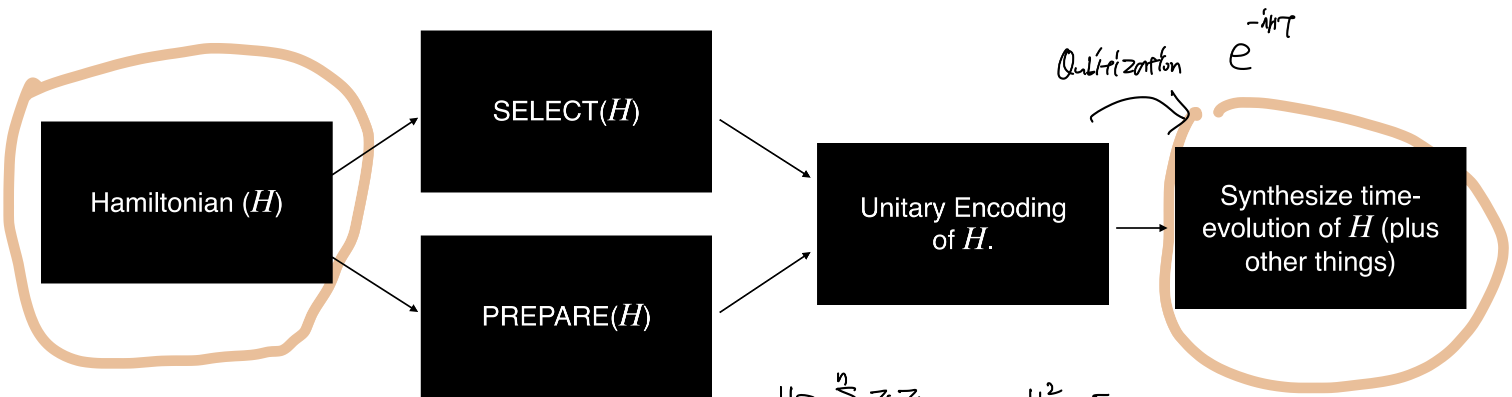
Overview

$$e^{-iHT} = \left(e^{-iHT/n}\right)^n \qquad \left\|\frac{HT}{n}\right\| = O(1)$$

$$H = \sum_P \alpha_P P$$

- LCU
  - ❖ Taylor expansion → Linear combination of unitary (Paulis)
  - ❖ SELECT: Selects the Pauli in the Taylor expansion
  - ❖ PREPARE: Encodes coefficients in the Taylor expansion
- Qubitization is an upgraded version of LCU, but it is more flexible and efficient.

High-level overview of Qubitization



$H = \sum_{i=1}^{n} Z_i Z_{i+1}$ $\quad$ $H^2 = \sum \cdots$

$Z_1 Z_6 \quad Z_1 Z_2 \quad Z_6 Z_6 \quad \ldots$

- ❖ SELECT: Selects the Pauli of the **Hamiltonian.**
- ❖ PREPARE: Encodes **coefficients of the Hamiltonian.**

Last time

$e^{-iHt} = \sum_P \alpha_P P$ $\quad$ $\mathrm{SELECT}|P\rangle|\psi\rangle = |P\rangle P|\psi\rangle$ (for P s.t. $\alpha_P \neq 0$)

$\mathrm{PREPARE}|0..0\rangle = \sum_P \sqrt{\alpha_P}\,|P\rangle$

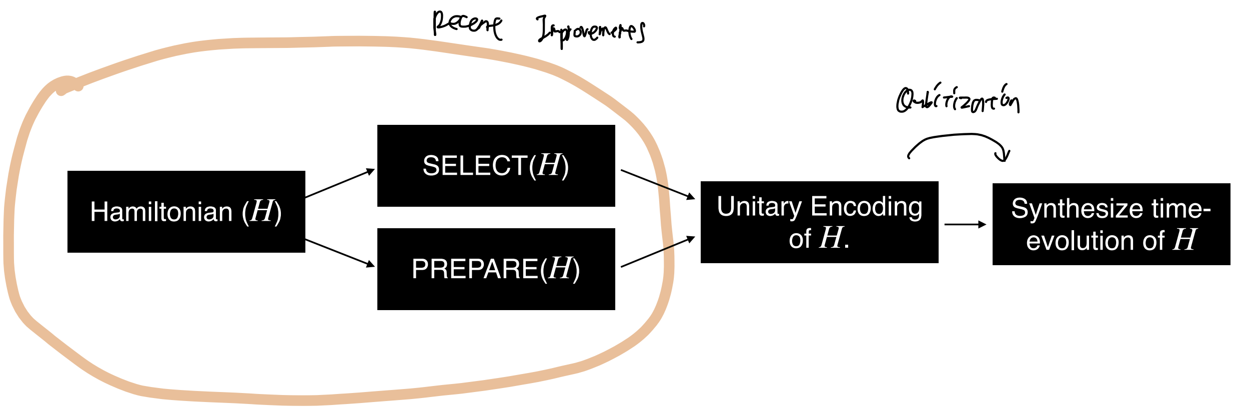
This lecture

$H = \sum_P \alpha'_P P$ $\quad$ $\mathrm{SELECT}(H)|P\rangle|\psi\rangle = |P\rangle P|\psi\rangle$ (for P s.t. $\alpha'_P \neq 0$)

$\mathrm{PREPARE}(H)|0\cdots0\rangle = \sum_P \sqrt{\alpha'_P}\,|P\rangle$

Why Qubitization?

- The best state-of-the-art quantum algorithms for quantum chemistry is based on qubitization, in one form or another.
- Improvements are made in more efficient constructions of SELECT and PREPARE, but the framework of qubitization lives on.
- If you view SELECT and PREPARE as black boxes, qubitization is optimal, and even the constants are reasonable.



Why Qubitization?

Year	Reference	Primary algorithmic innovation	Space complexity	Toffoli/T complexity
2005	Aspuru-Guzik *et al.* [4]	First algorithm (no compilation or bounds)	$\mathcal{O}(N)$	$\mathcal{O}(\text{poly}(N/\epsilon))$
2010	Whitfield *et al.* [11]	First compilation (no Trotter bounds)	$\mathcal{O}(N)$	$\mathcal{O}(\text{poly}(N/\epsilon))$
2012	Seeley *et al.* [41]	Use of Bravyi-Kitaev transformation	$\mathcal{O}(N)$	$\mathcal{O}(\text{poly}(N/\epsilon))$
2013	Wecker *et al.* [42]	First chemistry specific Trotter bounds	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(N^{10}/\epsilon^{3/2})$
2013	Toloui *et al.* [43]	Use of first quantization	$\mathcal{O}(\eta \log N)$	$\widetilde{\mathcal{O}}(\eta^2 N^8/\epsilon^{3/2})$
2014	Hastings *et al.* [44]	Better compilation and multi-resolution Trotter	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(N^8/\epsilon^{3/2})$
2014	Poulin *et al.* [45]	Tighter Trotter bounds and ordering	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(N^6/\epsilon^{3/2})$
2014	McClean *et al.* [25]	Exploiting Hamiltonian sparsity with Trotter	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(N^4 S/\epsilon^{3/2})$
2014	Babbush *et al.* [46]	Tighter system specific Trotter bounds	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(N^2 S/\epsilon^{3/2})$
2015	Babbush *et al.* [47]	Use of Taylor series (database method)	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(N^4 \lambda_V/\epsilon)$
2015	Babbush *et al.* [47]	Use of Taylor series (on-the-fly method)	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(N^5/\epsilon)$
2015	Babbush *et al.* [48]	Use of Taylor series with first quantization	$\mathcal{O}(\eta \log N)$	$\widetilde{\mathcal{O}}(\eta^2 N^3/\epsilon)$
2016	Reiher *et al.* [23]	First T count and tighter Trotter bounds	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(N^2 S/\epsilon^{3/2})$
2018	Motta *et al.* [29]	Use of low rank factorization with Trotter	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(N^4 \Xi/\epsilon^{3/2})$
2018	Campbell [49]	Use of randomized compiling with Trotter	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(\lambda_V^2/\epsilon^2)$
2019	Berry *et al.* [9]	Use of qubitization (sparse method)	$\widetilde{\mathcal{O}}(N+\sqrt{S})$	$\widetilde{\mathcal{O}}((N+\sqrt{S})\lambda_V/\epsilon)$
2019	Berry *et al.* [9]	Use of qubitization (single factorization)	$\widetilde{\mathcal{O}}(N^{3/2})$	$\widetilde{\mathcal{O}}(N^{3/2}\lambda_{\text{SF}}/\epsilon)$
2019	Kivlichan *et al.* [50]	Better randomized compiled phase estimation	$\mathcal{O}(N)$	$\widetilde{\mathcal{O}}(\lambda_V^2/\epsilon^2)$
2020	von Burg *et al.* [10]	Use of qubitization (double factorization)	$\widetilde{\mathcal{O}}(N\sqrt{\Xi})$	$\widetilde{\mathcal{O}}(N\lambda_{\text{DF}}\sqrt{\Xi}/\epsilon)$
2020	Present work	Use of tensor hypercontraction	$\widetilde{\mathcal{O}}(N)$	$\widetilde{\mathcal{O}}(N\lambda_\zeta/\epsilon)$

↳ qubitization

- From Lee et al., PRX Quantum 2, 030305 (2021).

Why Qubitization?

e^{-iHT}

- Qubitization is also **flexible**. While Hamiltonian simulation is one of its most important applications, it can also do other things.
- For instance, you can apply $\underbrace{e^{i\cos^{-1}(H/\|H\|)}}$. This may seem very obscure, but this is actually a very useful thing to do. (We'll talk about that next week.)

Quantum Signal Processing

- The precursor to qubitization is Quantum Signal Processing (QSP) [Low, Yoder, and Chuang (2016), Low and Chuang (2016)]
- Physical model: Always-on magnetic field in one direction + instantaneous pulses

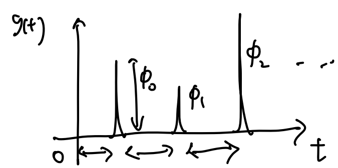
$$e^{i\phi_0 Z} e^{i\theta X} e^{i\phi_1 Z} \cdots e^{i\theta X} e^{i\phi_d Z}$$

$|0\rangle, |1\rangle$

$H = B_x X + g(t) Z$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$XY = iZ$ $YZ = iX$ $ZX = iY$



Quantum Signal Processing

- The precursor to qubitization is Quantum Signal Processing (QSP)
- Physical model: Always-on magnetic field in one direction + instantaneous pulses

d+1 variable angles ≈ sine

$$\underline{e^{i\phi_0 Z} e^{i\theta X} e^{i\phi_1 Z} \cdots e^{i\theta X} e^{i\phi_d Z}} = \begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix},$$

where $\theta = \cos^{-1}(a)$ and $a = \cos\theta$

1. Degrees of P and Q are at most d and d-1, respectively.
2. P, Q has parity d and (d-1) mod 2.
3. $|P|^2 + (1-a^2)|Q|^2 = 1$.

$$\begin{pmatrix} a\,e^{i(\phi_0+\phi_1)} & i\sqrt{1-a^2}\,e^{i(\phi_0-\phi_1)} \\ i\sqrt{1-a^2}\,e^{i(\phi_1-\phi_0)} & a\,e^{-i(\phi_0+\phi_1)} \end{pmatrix}$$

Quantum Signal Processing

$* \; e^{i\theta X} = I\cos\theta + i\sin\theta\, X$

- The precursor to qubitization is Quantum Signal Processing (QSP)
- Physical model: Always-on magnetic field in one direction + instantaneous pulses

$$\underbrace{e^{i\phi_0 Z} e^{i\theta X} e^{i\phi_1 Z}} = \begin{pmatrix} e^{i\phi_0} & 0 \\ 0 & e^{-i\phi_0} \end{pmatrix} \begin{pmatrix} a & i\sqrt{1-a^2} \\ i\sqrt{1-a^2} & a \end{pmatrix} \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_1} \end{pmatrix}$$

$$= \begin{pmatrix} a\, e^{i(\phi_0+\phi_1)} & i\sqrt{1-a^2}\, e^{i(\phi_0-\phi_1)} \\ i\sqrt{1-a^2}\, e^{i(\phi_1-\phi_0)} & a\, e^{-i(\phi_0+\phi_1)} \end{pmatrix}$$

where $\theta = \cos^{-1}(a)$.

1. Degrees of P and Q are at most d and d-1, respectively.
2. P, Q has parity d and (d-1) mod 2.
3. $|P|^2 + (1-a^2)|Q|^2 = 1.$

Quantum Signal Processing

- The precursor to qubitization is Quantum Signal Processing (QSP)
- Physical model: Always-on magnetic field in one direction + instantaneous pulses

$P_1 = P_1(a)$

$Q_1 = Q_1(a)$

$$e^{\frac{i\phi_1}{2}Z} e^{\frac{i\phi_1}{2}Z}$$

$$e^{i\phi_0 Z} e^{i\theta X} e^{i\phi_1 Z} e^{i\theta X} e^{i\phi_2 Z}$$

where $\theta = \cos^{-1}(a)$.

$$= \begin{pmatrix} P_1 & i\sqrt{1-a^2}\,Q_1 \\ i\sqrt{1-a^2}\,Q_1^* & P_1^* \end{pmatrix} \begin{pmatrix} P_2 & i\sqrt{1-a^2}\,Q_2 \\ i\sqrt{1-a^2}\,Q_2^* & P_2^* \end{pmatrix} = \begin{pmatrix} P_1P_2 - (1-a^2)Q_1Q_2 & i\sqrt{1-a^2}\,(P_1Q_2 + Q_1P_2^*) \\ - - & - - \end{pmatrix}$$

1. Degrees of P and Q are at most d and d-1, respectively.
2. P, Q has parity d and (d-1) mod 2.
3. $|P|^2 + (1-a^2)|Q|^2 = 1.$

$$\begin{pmatrix} a\,e^{i(\phi_0+\phi_1)} & i\sqrt{1-a^2}\,e^{i(\phi_0-\phi_1)} \\ i\sqrt{1-a^2}\,e^{i(\phi_1-\phi_0)} & a\,e^{-i(\phi_0+\phi_1)} \end{pmatrix}$$

	degree	even/odd
P_1:	1	odd
Q_1:	0	even
P_2:	1	odd
Q_2:	0	even

Recursion

$$e^{i\phi_1 Z/2} e^{i\phi_1 Z/2}$$

$$e^{i\phi_0 Z} e^{i\theta X} e^{i\phi_1 Z} \cdots e^{i\theta X} e^{i\phi_d Z} = \begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix},$$

where $\theta = \cos^{-1}(a)$ and

1. Degrees of P and Q are at most d and d-1, respectively.
2. P, Q has parity d and (d-1) mod 2.
3. $|P|^2 + (1-a^2)|Q|^2 = 1.$

$$\begin{pmatrix} P_1 & i\sqrt{1-a^2}\,Q_1 \\ i\sqrt{1-a^2}\,Q_1^* & P_1^* \end{pmatrix} \begin{pmatrix} P_2 & i\sqrt{1-a^2}\,Q_2 \\ i\sqrt{1-a^2}\,Q_2^* & P_2^* \end{pmatrix} = \begin{pmatrix} P_1P_2 - (1-a^2)Q_1Q_2 & i\sqrt{1-a^2}\,(P_1Q_2 + Q_1P_2^*) \\ \text{- -} & \text{- -} \end{pmatrix}$$

	degree	even/odd
P_1:	1	odd
Q_1:	0	even
P_2:	1	odd
Q_2:	0	even

	degree
P_1	d_1
Q_1	d_1-1
P_2	d_2
Q_2	d_2-1

$$\deg(P_1P_2 + (1-a^2)Q_1Q_2) = d_1 + d_2$$

$$\deg(P_1Q_2 + Q_1P_2^*) = d_1 + d_2 - 1$$

A remarkable fact: Converse is true!

- For any

$$\begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix},$$

where $\theta = \cos^{-1}(a)$ and

1. Degrees of P and Q are at most d and d-1, respectively.
2. P, Q has parity d and (d-1) mod 2.
3. $|P|^2 + (1-a^2)|Q|^2 = 1.$

There is a set of angles $\phi_0, \ldots \phi_d$ such that $e^{i\phi_0 Z} e^{i\theta X} e^{i\phi_1 Z} \ldots e^{i\theta X} e^{i\phi_d Z} = \begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix}.$

An analogy

$e^{-iZ\phi_0}$

$Z(\phi_0)$ $X(\theta)$ $Z(\phi_1)$ $X(\theta)$ $Z(\phi_2)$ $X(\theta)$ $Z(\phi_3)$: Quantum signal processing

$e^{-iZ\phi_0}$

$Z(\phi_0)$ $U(H)$ $Z(\phi_1)$ $U(H)$ $Z(\phi_2)$ $U(H)$ $Z(\phi_3)$: Qubitization

$$U(H) = \begin{array}{cc} \overbrace{}^{N} & \overbrace{}^{N} \\ \end{array}$$

$$U(H) = \begin{pmatrix} H/\|H\| & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{array}{l} \}N \\ \}N \end{array}$$

$\|U(H)\| = 1$

Recap

- Quantum Signal Processing: A flexible framework to synthesize an arbitrary element of $SU(2)$ by an always-on magnetic field and a sequence of variable “pulses.”
- This is just about a qubit, but the lesson extends more generally. Let’s talk about that in the next lecture.